

# Package ‘QZ’

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**Title** Generalized Eigenvalues and QZ Decomposition

**Depends** R (>= 3.6.0), methods, Matrix

**Suggests** fda

**LazyLoad** yes

**LazyData** yes

**Copyright** See QZ/inst/LAPACK\_LICENSE.txt for the files in src/qz/.

**Description** Generalized eigenvalues and eigenvectors  
use QZ decomposition (generalized Schur decomposition).  
The decomposition needs an N-by-N non-symmetric  
matrix A or paired matrices (A,B) with eigenvalues reordering  
mechanism. The decomposition functions are mainly based Fortran  
subroutines in complex\*16 and double precision of LAPACK  
library (version 3.10.0 or later).

**License** Mozilla Public License 2.0

**NeedsCompilation** yes

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QZ-package	<i>Generalized Eigenvalues and QZ Decomposition</i>
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## Description

QZ package provides generalized eigenvalues and QZ decomposition (generalized Schur form) for an N-by-N non-symmetric matrix A or paired matrices (A,B) with eigenvalues reordering mechanism. The package is mainly based complex\*16 and double precision of LAPACK library (version 3.4.2.)

## Details

Package:	QZ
Type:	Package
License:	MPL
LazyLoad:	yes

The QZ package contains R functions for generalized eigenvalues and QZ decomposition (generalized Schur form) for an N-by-N non-symmetric matrix A or paired matrices (A,B) via two main functions, `qz.geigen()` and `qz()`. The `qz()` function also provides an option for eigenvalues reordering.

The QZ package is also based on a minimum set of complex\*16 and double precision of LAPACK and BLAS Fortran libraries. Most functions are wrapped in C via `.Call()` to avoid extra memory copy and to improve performance and memory usage.

## Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

<https://www.netlib.org/lapack/>

**See Also**

[qz.geigen](#), [qz](#),

[qz.zgges](#), [qz.zggeev](#), [qz.ztgsen](#), [qz.dgges](#), [qz.dggeev](#), [qz.dtgsen](#),

[qz.zgees](#), [qz.zgeev](#), [qz.ztrsen](#), [qz.dgees](#), [qz.dgeev](#), [qz.dtrsen](#).

**Examples**

```
## Not run:
demo(ex1_geigen, "QZ")
demo(ex2_qz, "QZ")
demo(ex3_ordqz, "QZ")
demo(ex4_fda_geigen, "QZ")

## End(Not run)
```

---

Conjugate transpose      *Conjugate Transpose for Complex Matrix*

---

**Description**

Conjugate transpose, Hermitian transpose, or Hermitian conjugate.

**Usage**

```
H(x)
```

**Arguments**

x                    a complex matrix or vector.

**Details**

This is equivalent to `Conj(t.default(x))`.

**Value**

This returns a conjugate transpose of x.

**Author(s)**

Wei-Chen Chen <[wccsnow@gmail.com](mailto:wccsnow@gmail.com)>

**Examples**

```
library(QZ, quiet = TRUE)

A <- matrix(c(-21.10 -22.50i, 53.50 -50.50i, -34.50 +127.50i, 7.50 +0.50i,
             -0.46 -7.78i, -3.50 -37.50i, -15.50 +58.50i, -10.50 -1.50i,
             4.30 -5.50i, 39.70 -17.10i, -68.50 +12.50i, -7.50 -3.50i,
             5.50 +4.40i, 14.40 +43.30i, -32.50 -46.00i, -19.00 -32.50i),
            nrow = 4, byrow = TRUE)

H(A)
```

---

 Example datasets

*Small example datasets*


---

**Description**

These datasets are small for test operations and functions in complex and double precision/matrices.

**Format**

Each dataset contains information where it is from and two matrices in pair of (A,B) or single matrix (A) for testing functions `qz.*` or related functions, either in complex or in double precision.

**Details**

The example datasets are

Examples	Source
exAB1	<a href="https://www.nag.com/lapack-ex/node124.html">https://www.nag.com/lapack-ex/node124.html</a>
exAB2	<a href="https://www.nag.com/lapack-ex/node119.html">https://www.nag.com/lapack-ex/node119.html</a>
exAB3	<a href="https://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/fl08/fl08yuf.xml">https://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/fl08/fl08yuf.xml</a>
exAB4	<a href="https://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/fl08/fl08ygf.xml">https://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/fl08/fl08ygf.xml</a>
exA1	<a href="https://www.nag.com/lapack-ex/node94.html">https://www.nag.com/lapack-ex/node94.html</a>
exA2	<a href="https://www.nag.com/lapack-ex/node89.html">https://www.nag.com/lapack-ex/node89.html</a>
exA3	<a href="https://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/fl08/fl08quf.xml">https://www.nag.com/numeric/fl/nagdoc_fl23/xhtml/fl08/fl08quf.xml</a>
exA4	<a href="https://www.nag.com/numeric/fl/nagdoc_fl22/xhtml/fl08/fl08qgf.xml">https://www.nag.com/numeric/fl/nagdoc_fl22/xhtml/fl08/fl08qgf.xml</a>

The elements of dataset are (if any)

Elements	Usage
description	the source of data
A	the first matrix A
B	the second matrix B
S	the Shur form

T	the Shur form
Q	the left Shur vectors
Z	the right Shur vectors

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

---

fda.geigen

*Generalized Eigen Analysis as in fda Package*

---

**Description**

This is an equivalent function to `fda::geigen` which finds matrices  $L$  and  $M$  to maximize  $\text{tr}(L'AM) / \sqrt{\text{tr}(L'BL) \text{tr}(M'CM)}$

where  $A = a \text{ } p \times q$  matrix,  $B = p \times p$  symmetric, positive definite matrix,  $C = q \times q$  symmetric positive definite matrix,  $L = p \times s$  matrix, and  $M = q \times s$  matrix, where  $s =$  the number of non-zero generalized eigenvalues of  $A$ .

**Usage**

```
fda.geigen(Amat, Bmat, Cmat)
```

**Arguments**

Amat	a numeric matrix
Bmat	a symmetric, positive definite matrix with dimension = number of rows of A
Cmat	a symmetric, positive definite matrix with dimension = number of columns of A

**Details**

This function is equivalent to `fda::geigen(Amat,Bmat,Cmat)` except that this is rewritten and utilizes LAPACK functions via `qz.dggev`.

Also, `Lmat` and `Mmat` are both scaled such that  $L'BL$  and  $M'CM$  are identity matrices.

**Value**

`list(values, Lmat, Mmat)`

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**See Also**

[qz.geigen](#), [qz.dggev](#).

**Examples**

```
library(QZ, quiet = TRUE)

A <- matrix(as.double(1:6), 2)
B <- matrix(as.double(c(2, 1, 1, 2)), 2)
C <- diag(as.double(1:3))

ret.qz <- fda.geigen(A, B, C)

### Verify
library(fda, quiet = TRUE)
ret.fda <- fda::geigen(A, B, C)
```

---

Generalized Eigenvalues

*Generalized Eigen Values*

---

**Description**

This function obtains generalized eigen values on input paired matrices (A,B) or a single matrix A.

**Usage**

```
geigen(A, B = NULL, only.values = FALSE, ...)

qz.geigen(A, B = NULL, only.values = FALSE, ...)
```

**Arguments**

A	a 'complex/real' matrix, dim = c(N, N).
B	a 'complex/real' matrix, dim = c(N, N).
only.values	if 'TRUE', only the eigenvalues are computed and returned, otherwise both eigenvalues and eigenvectors are returned.
...	options to qz.* functions.

**Details**

Call one of [qz.zggeev](#), [qz.dggeev](#), [qz.zgeev](#), or [qz.dgeev](#) depending on the input arguments and types.

**Value**

Returns a list from the call.

**Author(s)**

Wei-Chen Chen <[wccsnow@gmail.com](mailto:wccsnow@gmail.com)>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

[qz](#), [ordqz](#).

**Examples**

```
library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node122.html
(ret <- qz.geigen(exAB1$A, exAB1$B))

### https://www.nag.com/lapack-ex/node117.html
(ret <- qz.geigen(exAB2$A, exAB2$B))

### https://www.nag.com/lapack-ex/node92.html
(ret <- qz.geigen(exA1$A))

### https://www.nag.com/lapack-ex/node87.html
(ret <- qz.geigen(exA2$A))
```

**Description**

Several classes are declared in **QZ**, and these are functions to print objects.

**Usage**

```

## S3 method for class 'zgges'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'zggev'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'ztgsen'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'dgges'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'dggev'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'dtgsen'
print(x, digits = max(4, getOption("digits") - 3), ...)

## S3 method for class 'zgees'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'zgeev'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'ztrsen'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'dgees'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'dgeev'
print(x, digits = max(4, getOption("digits") - 3), ...)
## S3 method for class 'dtrsen'
print(x, digits = max(4, getOption("digits") - 3), ...)

```

**Arguments**

<code>x</code>	an object with the class attributes.
<code>digits</code>	for printing out numbers.
<code>...</code>	other possible options.

**Details**

These are useful functions for summarizing and debugging. Use `names` or `str` to explore the details.

**Value**

The results will cat or print on the STDOUT by default.

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.  
[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)



**See Also**

[qz.zgges](#), [qz.zggeev](#), [qz.ztgsgen](#), [qz.dgges](#), [qz.dggeev](#), [qz.dtgsgen](#),  
[qz.zgees](#), [qz.zgeev](#), [qz.ztrsgen](#), [qz.dgees](#), [qz.dgeev](#), [qz.dtrsgen](#).

**Examples**

```
## Not run:
# Functions applied by directly type the names of objects.

## End(Not run)
```

---

QZ Decomposition

*QZ Decomposition*

---

**Description**

This function performs QZ decomposition on input paired matrices (A,B) or a single matrix A.

**Usage**

```
qz(A, B = NULL, select = NULL, only.values = FALSE, ...)
```

**Arguments**

A	a 'complex/real' matrix, dim = c(N, N).
B	a 'complex/real' matrix, dim = c(N, N).
select	specifies the eigenvalues in the selected cluster.
only.values	if 'TRUE', only the eigenvalues are computed and returned, otherwise both eigenvalues and eigenvectors are returned.
...	options to qz.* functions.

**Details**

If select is NULL, then call one of [qz.zgges](#), [qz.dgges](#), [qz.zgees](#), or [qz.dgees](#) depending on the input arguments and types.

If select is not NULL, then call one of [qz.zgges](#) + [qz.ztgsgen](#), [qz.dgges](#) + [qz.dtgsgen](#), [qz.zgees](#) + [qz.ztrsgen](#), or [qz.dgees](#) + [qz.dtrsgen](#) depending on the input arguments and types.

**Value**

Returns a list from the call.

**Author(s)**

Wei-Chen Chen <[wccsnow@gmail.com](mailto:wccsnow@gmail.com)>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.  
[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

[ordqz](#), [geigen](#).

**Examples**

```
library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node124.html
(ret <- qz(exAB1$A, exAB1$B))

### https://www.nag.com/lapack-ex/node119.html
(ret <- qz(exAB2$A, exAB2$B))

### https://www.nag.com/lapack-ex/node94.html
(ret <- qz(exA1$A))

### https://www.nag.com/lapack-ex/node89.html
(ret <- qz(exA2$A))

# Reordering eigenvalues
select1 <- c(TRUE, FALSE, FALSE, TRUE)
select2 <- c(FALSE, TRUE, TRUE, FALSE)
(ret <- qz(exAB1$A, exAB1$B, select = select1))
(ret <- qz(exAB2$A, exAB2$B, select = select2))
(ret <- qz(exA1$A, select = select1))
(ret <- qz(exA2$A, select = select1))
```

---

QZ Decomposition Reordering

*Reordering QZ Decomposition*

---

**Description**

This function performs QZ decomposition on input paired matrices (A,B) or a single matrix A with reordering.

**Usage**

```
ordqz(A, B = NULL, cluster = NULL,
      keyword = c("lhp", "rhp", "udi", "udo", "ref", "cef",
                 "lhp.fo", "rhp.fo", "udi.fo", "udo.fo"),
      ...)
```

**Arguments**

A	a 'complex/real' matrix, dim = c(N, N).
B	a 'complex/real' matrix, dim = c(N, N).
cluster	specifies the eigenvalues in the selected cluster.
keyword	as similarly used in MATLAB.
...	options to qz.* functions.

**Details**

Either cluster or keyword should be specified.

cluster actually is the same as select in all qz.\* functions.

keyword actually is similar as MATLAB.

keyword	Selected Region
'lhp'	Left-half plane ( $\text{real}(E) < 0$ )
'rhp'	Right-half plane ( $\text{real}(E) \geq 0$ )
'udi'	Interior of unit disk ( $\text{abs}(E) < 1$ )
'udo'	Exterior of unit disk ( $\text{abs}(E) \geq 1$ )
'ref'	Real eigenvalues first (top-left corner)
'cef'	Complex eigenvalues first (top-left corner)
'lhp'	Left-half plane ( $\text{real}(E) < 0$ ) and finite only
'rhp'	Right-half plane ( $\text{real}(E) \geq 0$ ) and finite only
'udi'	Interior of unit disk ( $\text{abs}(E) < 1$ ) and finite only
'udo'	Exterior of unit disk ( $\text{abs}(E) \geq 1$ ) and finite only

**Value**

Returns a list from the call.

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

qz, geigen.

**Examples**

```
library(QZ, quiet = TRUE)
```

```
# Reordering eigenvalues
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "lhp"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "rhp"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "udi"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "udo"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "ref"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "cef"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "lhp.fo"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "rhp.fo"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "udi.fo"))
(ret <- ordqz(exAB1$A, exAB1$B, keyword = "udo.fo"))
```

---

qz.dgees

*QZ Decomposition for a Real Matrix*


---

### Description

This function call 'dgees' in Fortran to decompose a 'real' matrix A.

### Usage

```
qz.dgees(A, vs = TRUE, LWORK = NULL)
```

### Arguments

A	a 'real' matrix, dim = c(N, N).
vs	if compute 'real' Schur vectors. (Q)
LWORK	optional, dimension of array WORK for workspace. (>= 3N)

### Details

See 'dgees.f' for all details.

DGEES computes for an N-by-N real non-symmetric matrix A, the eigenvalues, the real Schur form T, and, optionally, the matrix of Schur vectors Q. This gives the Schur factorization  $A = Q^*T^*(Q^{**}T)$ .

Optionally, it also orders the eigenvalues on the diagonal of the real Schur form so that selected eigenvalues are at the top left. The leading columns of Q then form an orthonormal basis for the invariant subspace corresponding to the selected eigenvalues.

A matrix is in real Schur form if it is upper quasi-triangular with 1-by-1 and 2-by-2 blocks. 2-by-2 blocks will be standardized in the form

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix}$$

where  $b^*c < 0$ . The eigenvalues of such a block are  $a \pm \sqrt{bc}$ .

**Value**

Return a list contains next:

'T'	A's generalized Schur form.
'WR'	original returns from 'dgees.f'.
'WI'	original returns from 'dgees.f'.
'VS'	original returns from 'dgees.f'.
'WORK'	optimal LWORK (for dgees.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. <= N: QZ iteration failed. =N+1: reordering problem. =N+2: reordering failed.

Extra returns in the list:

'W'	WR + WI * i.
'Q'	the Schur vectors.

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/double/dgees.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

[qz.dgeev](#)

**Examples**

```
library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node89.html
A <- exA2$A
ret <- qz.dgees(A)

# Verify 1
A.new <- ret$Q %*% ret$T %*% solve(ret$Q)
round(A - A.new)

# verify 2
round(ret$Q %*% solve(ret$Q))
```

qz.dgeev

*Generalized Eigenvalues Decomposition for a Real Matrix***Description**

This function call 'dgeev' in Fortran to decompose a 'real' matrix A.

**Usage**

```
qz.dgeev(A, vl = TRUE, vr = TRUE, LWORK = NULL)
```

**Arguments**

A	a 'real' matrix, dim = c(N, N).
vl	if compute left 'real' eigen vector. (U)
vr	if compute right 'real' eigen vector. (V)
LWORK	optional, dimension of array WORK for workspace. (>= 4N)

**Details**

See 'dgeev.f' for all details.

DGEEV computes for an N-by-N real non-symmetric matrix A, the eigenvalues and, optionally, the left and/or right eigenvectors.

The right eigenvector v(j) of A satisfies

$$A * v(j) = \text{lambda}(j) * v(j)$$

where lambda(j) is its eigenvalue. The left eigenvector u(j) of A satisfies

$$u(j)**T * A = \text{lambda}(j) * u(j)**T$$

where u(j)\*\*T denotes the transpose of u(j).

The computed eigenvectors are normalized to have Euclidean norm equal to 1 and largest component real.

**Value**

Return a list contains next:

'WR'	original returns from 'dgeev.f'.
'WI'	original returns from 'dgeev.f'.
'VL'	original returns from 'dgeev.f'.
'VR'	original returns from 'dgeev.f'.
'WORK'	optimal LWORK (for dgeev.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. > 0: QZ iteration failed.

Extra returns in the list:

'W'                WR + WI \* i.  
 'U'                the left eigen vectors.  
 'V'                the right eigen vectors.

If WI[j] is zero, then the j-th eigenvalue is real; if positive, then the j-th and (j+1)-st eigenvalues are a complex conjugate pair, with WI[j+1] negative.

If the j-th eigenvalue is real, then U[, j] = VL[, j], the j-th column of VL. If the j-th and (j+1)-th eigenvalues form a complex conjugate pair, then U[, j] = VL[, j] + i \* VL[, j+1] and U[, j+1] = VL[, j] - i \* VL[, j+1].

Similarly, for the right eigenvectors of V and VR.

### Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

### References

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/double/dgeev.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

### See Also

[qz.dgees](#)

### Examples

```
library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node87.html
A <- exA2$A
ret <- qz.dgeev(A)

# Verify 1
diff.R <- A %*% ret$V - matrix(ret$W, 4, 4, byrow = TRUE) * ret$V
diff.L <- t(ret$U) %*% A - matrix(ret$W, 4, 4) * t(ret$U)
round(diff.R)
round(diff.L)

# Verify 2
round(ret$U %*% solve(ret$U))
round(ret$V %*% solve(ret$V))
```

qz.dgges

*QZ Decomposition for Real Paired Matrices***Description**

This function call 'dgges' in Fortran to decompose 'real' matrices (A,B).

**Usage**

```
qz.dgges(A, B, vs1 = TRUE, vsr = TRUE, LWORK = NULL)
```

**Arguments**

A	a 'real' matrix, dim = c(N, N).
B	a 'real' matrix, dim = c(N, N).
vs1	if compute left 'real' Schur vectors. (Q)
vsr	if compute right 'real' Schur vectors. (Z)
LWORK	optional, dimension of array WORK for workspace. ( $\geq 8N+16$ )

**Details**

See 'dgges.f' for all details.

DGGEES computes for a pair of N-by-N real non-symmetric matrices (A,B), the generalized eigenvalues, the generalized real Schur form (S,T), optionally, the left and/or right matrices of Schur vectors (VSL and VSR). This gives the generalized Schur factorization

$$(A,B) = ( (VSL)*S*(VSR)**T, (VSL)*T*(VSR)**T )$$

Optionally, it also orders the eigenvalues so that a selected cluster of eigenvalues appears in the leading diagonal blocks of the upper quasi-triangular matrix S and the upper triangular matrix T. The leading columns of VSL and VSR then form an orthonormal basis for the corresponding left and right eigenspaces (deflating subspaces).

(If only the generalized eigenvalues are needed, use the driver DGGEV instead, which is faster.)

A generalized eigenvalue for a pair of matrices (A,B) is a scalar w or a ratio  $\alpha/\beta = w$ , such that  $A - w*B$  is singular. It is usually represented as the pair (alpha,beta), as there is a reasonable interpretation for  $\beta=0$  or both being zero.

A pair of matrices (S,T) is in generalized real Schur form if T is upper triangular with non-negative diagonal and S is block upper triangular with 1-by-1 and 2-by-2 blocks. 1-by-1 blocks correspond to real generalized eigenvalues, while 2-by-2 blocks of S will be "standardized" by making the corresponding elements of T have the form:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and the pair of corresponding 2-by-2 blocks in S and T will have a complex conjugate pair of generalized eigenvalues.



**Value**

Return a list contains next:

'S'	A's generalized Schur form.
'T'	B's generalized Schur form.
'ALPHAR'	original returns from 'dgges.f'.
'ALPHAI'	original returns from 'dgges.f'.
'BETA'	original returns from 'dgges.f'.
'VSL'	original returns from 'dgges.f'.
'VSR'	original returns from 'dgges.f'.
'WORK'	optimal LWORK (for dgges.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1,...,N: QZ iteration failed. =N+1: other than QZ iteration failed in DHGEQZ. =N+2: reordering problem. =N+3: reordering failed.

Extra returns in the list:

'ALPHA'	ALPHAR + ALPHAI * i.
'Q'	the left Schur vectors.
'Z'	the right Schur vectors.

The ALPHA[j]/BETA[j] are generalized eigenvalues.

If ALPHAI[j] is zero, then the j-th eigenvalue is real; if positive, then the j-th and (j+1)-st eigenvalues are a complex conjugate pair, with ALPHAI[j+1] negative.

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/double/dgges.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

[qz.dggev](#)

**Examples**

```

library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node119.html
A <- exAB2$A
B <- exAB2$B
ret <- qz.dgges(A, B)

# Verify 1
A.new <- ret$Q %*% ret$S %*% t(ret$Z)
B.new <- ret$Q %*% ret$T %*% t(ret$Z)
round(A - A.new)
round(B - B.new)

# verify 2
round(ret$Q %*% t(ret$Q))
round(ret$Z %*% t(ret$Z))

```

---

qz.dggeev

*Generalized Eigenvalues Decomposition for Real Paired Matrices*


---

**Description**

This function call 'dggev' in Fortran to decompose 'real' matrices (A,B).

**Usage**

```
qz.dggeev(A, B, vl = TRUE, vr = TRUE, LWORK = NULL)
```

**Arguments**

A	a 'real' matrix, dim = c(N, N).
B	a 'real' matrix, dim = c(N, N).
vl	if compute left 'real' eigen vector. (U)
vr	if compute right 'real' eigen vector. (V)
LWORK	optional, dimension of array WORK for workspace. (>= 8N)

**Details**

See 'dggev.f' for all details.

DGGEV computes for a pair of N-by-N real non-symmetric matrices (A,B) the generalized eigenvalues, and optionally, the left and/or right generalized eigenvectors.

A generalized eigenvalue for a pair of matrices (A,B) is a scalar lambda or a ratio alpha/beta = lambda, such that A - lambda\*B is singular. It is usually represented as the pair (alpha,beta), as there is a reasonable interpretation for beta=0, and even for both being zero.

The right eigenvector  $v(j)$  corresponding to the eigenvalue  $\lambda(j)$  of (A,B) satisfies  $A * v(j) = \lambda(j) * B * v(j)$ .

The left eigenvector  $u(j)$  corresponding to the eigenvalue  $\lambda(j)$  of (A,B) satisfies  $u(j)**H * A = \lambda(j) * u(j)**H * B$ .

where  $u(j)**H$  is the conjugate-transpose of  $u(j)$ .

## Value

Return a list contains next:

'ALPHAR'	original returns from 'dggev.f'.
'ALPHAI'	original returns from 'dggev.f'.
'BETA'	original returns from 'dggev.f'.
'VL'	original returns from 'dggev.f'.
'VR'	original returns from 'dggev.f'.
'WORK'	optimal LWORK (for dggev.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1,...,N: QZ iteration failed. =N+1: other than QZ iteration failed in DHGEQZ. =N+2: reordering problem. =N+3: reordering failed.

Extra returns in the list:

'ALPHA'	ALPHAR + ALPHAI * i.
'U'	the left eigen vectors.
'V'	the right eigen vectors.

If ALPHAI[j] is zero, then the j-th eigenvalue is real; if positive, then the j-th and (j+1)-st eigenvalues are a complex conjugate pair, with ALPHAI[j+1] negative.

If the j-th eigenvalue is real, then  $U[, j] = VL[, j]$ , the j-th column of VL. If the j-th and (j+1)-th eigenvalues form a complex conjugate pair, then  $U[, j] = VL[, j] + i * VL[, j+1]$  and  $U[, j+1] = VL[, j] - i * VL[, j+1]$ . Each eigenvector is scaled so the largest component has  $\text{abs}(\text{real part}) + \text{abs}(\text{imag. part}) = 1$ .

Similarly, for the right eigenvectors of V and VR.

## Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

## References

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/double/dggeev.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

## See Also

[qz.dgges](#)

**Examples**

```

library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node117.html
A <- exAB2$A
B <- exAB2$B
ret <- qz.dggev(A, B)

# Verify
(lambda <- ret$ALPHA / ret$BETA) # Unstable
diff.R <- matrix(ret$BETA, 4, 4, byrow = TRUE) * A %% ret$V -
          matrix(ret$ALPHA, 4, 4, byrow = TRUE) * B %% ret$V
diff.L <- matrix(ret$BETA, 4, 4) * H(ret$U) %% A -
          matrix(ret$ALPHA, 4, 4) * H(ret$U) %% B
round(diff.R)
round(diff.L)

# Verify 2
round(ret$U %% solve(ret$U))
round(ret$V %% solve(ret$V))

```

qz.dtgsen

*Reordered QZ Decomposition for Real Paired Matrices***Description**

This function call 'dtgsend' in Fortran to reorder 'double' matrices (S,T,Q,Z).

**Usage**

```

qz.dtgsen(S, T, Q, Z, select, ijob = 4L,
          want.Q = TRUE, want.Z = TRUE, LWORK = NULL, LIWORK = NULL)

```

**Arguments**

S	a 'double' generalized Schur form, dim = c(N, N).
T	a 'double' generalized Schur form, dim = c(N, N).
Q	a 'double' left Schur vectors, dim = c(N, N).
Z	a 'double' right Schur vectors, dim = c(N, N).
select	specifies the eigenvalues in the selected cluster.
ijob	specifies whether condition numbers are required for the cluster of eigenvalues (PL and PR) or the deflating subspaces (Difu and Difl).
want.Q	if update Q.
want.Z	if update Z.
LWORK	optional, dimension of array WORK for workspace. ( $\geq \max(4N+16, N(N+1))$ )
LIWORK	optional, dimension of array IWORK for workspace. ( $\geq \max(N+6, N(N+1)/2$ )

## Details

See 'dtgsen.f' for all details.

DTGSEN reorders the generalized real Schur decomposition of a real matrix pair (S,T) (in terms of an orthonormal equivalence transformation  $Q^{**T} * (S,T) * Z$ ), so that a selected cluster of eigenvalues appears in the leading diagonal blocks of the upper quasi-triangular matrix S and the upper triangular T. The leading columns of Q and Z form orthonormal bases of the corresponding left and right eigenspaces (deflating subspaces). (S,T) must be in generalized real Schur canonical form (as returned by DGGES), i.e. S is block upper triangular with 1-by-1 and 2-by-2 diagonal blocks. T is upper triangular.

Note for 'ijob':

=0: Only reorder w.r.t. SELECT. No extras.

=1: Reciprocal of norms of "projections" onto left and right eigenspaces w.r.t. the selected cluster (PL and PR).

=2: Upper bounds on Difl and Difl. F-norm-based estimate (DIF(1:2)).

=3: Estimate of Difl and Difl. 1-norm-based estimate (DIF(1:2)). About 5 times as expensive as ijob = 2.

=4: Compute PL, PR and DIF (i.e. 0, 1 and 2 above): Economic version to get it all.

=5: Compute PL, PR and DIF (i.e. 0, 1 and 3 above).

In short, if  $(A,B) = Q * (S,T) * Z^{**T}$  from qz.zgges and input (S,T,Q,Z) to qz.ztgsen with appropriate select option, then it yields

$$(A,B) = Q\_n * (S\_n, T\_n) * Z\_n^{**T}$$

where  $(S\_n, T\_n, Q\_n, Z\_n)$  is a new set of generalized Schur decomposition of (A,B) according to the select.

## Value

Return a list contains next:

'S'	S's reorded generalized Schur form.
'T'	T's reorded generalized Schur form.
'ALPHAR'	original returns from 'dtgsen.f'.
'ALPHAI'	original returns from 'dtgsen.f'.
'BETA'	original returns from 'dtgsen.f'.
'M'	original returns from 'dtgsen.f'.
'PL'	original returns from 'dtgsen.f'.
'PR'	original returns from 'dtgsen.f'.
'DIF'	original returns from 'dtgsen.f'.
'WORK'	optimal LWORK (for dtgsen.f only)
'IWORK'	optimal LIWORK (for dtgsen.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1: reordering of (S,T) failed.

Extra returns in the list:

'ALPHA'          ALPHAR + ALPHAI \* i.  
 'Q'                the reorded left Schur vectors.  
 'Z'                the reorded right Schur vectors.

### Warning(s)

There is no format checking for S, T, Q, and Z which are usually returned by qz.dgges.

There is also no checking for select which is usually according to the returns of qz.dggev.

### Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

### References

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/double/dtgsen.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

### See Also

[qz.zgges](#), [qz.dgges](#), [qz.ztgsen](#).

### Examples

```
library(QZ, quiet = TRUE)

### https://www.nag.com/numeric/f1/nagdoc_f123/xhtml/f08/f08ygf.xml
S <- exAB4$$
T <- exAB4$t
Q <- exAB4$Q
Z <- exAB4$Z
select <- c(FALSE, TRUE, TRUE, FALSE)
ret <- qz.dtgsen(S, T, Q, Z, select)

# Verify 1
S.new <- ret$Q %*% ret$S %*% t(ret$Z)
T.new <- ret$Q %*% ret$T %*% t(ret$Z)
round(S - S.new)
round(T - T.new)

# verify 2
round(ret$Q %*% t(ret$Q))
round(ret$Z %*% t(ret$Z))
```

---

qz.dtrsens                      *Reordered QZ Decomposition for a Real Matrix*

---

### Description

This function call 'dtrsens' in Fortran to reorder 'double' matrices (T,Q).

### Usage

```
qz.dtrsens(T, Q, select, job = c("B", "V", "E", "N"),
          want.Q = TRUE, LWORK = NULL, LIWORK = NULL)
```

### Arguments

T	a 'double' generalized Schur form, dim = c(N, N).
Q	a 'double' Schur vectors, dim = c(N, N).
select	specifies the eigenvalues in the selected cluster.
job	Specifies whether condition numbers are required for the cluster of eigenvalues (S) or the invariant subspace (SEP).
want.Q	if update Q.
LWORK	optional, dimension of array WORK for workspace. ( $\geq N(N+1)/2$ )
LIWORK	optional, dimension of array IWORK for workspace. ( $\geq N(N+1)/4$ )

### Details

See 'dtrsens.f' for all details.

DTRSENS reorders the real Schur factorization of a real matrix  $A = Q^*T^*Q^{**}T$ , so that a selected cluster of eigenvalues appears in the leading diagonal blocks of the upper quasi-triangular matrix T, and the leading columns of Q form an orthonormal basis of the corresponding right invariant subspace.

Optionally the routine computes the reciprocal condition numbers of the cluster of eigenvalues and/or the invariant subspace.

T must be in Schur canonical form (as returned by DHSEQR), that is, block upper triangular with 1-by-1 and 2-by-2 diagonal blocks; each 2-by-2 diagonal block has its diagonal elements equal and its off-diagonal elements of opposite sign.

### Value

Return a list contains next:

'T'	T's reorded generalized Schur form.
'WR'	original returns from 'dtrsens.f'.
'WI'	original returns from 'dtrsens.f'.
'M'	original returns from 'dtrsens.f'.

'S' original returns from 'dtrsens.f'.  
 'SEP' original returns from 'dtrsens.f'.  
 'WORK' optimal LWORK (for dtrsens.f only)  
 'IWORK' optimal LIWORK (for dtrsens.f only)  
 'INFO' = 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1: reordering of T failed.

Extra returns in the list:

'W' WR + WI \* i.  
 'Q' the reorded Schur vectors.

### Warning(s)

There is no format checking for T and Q which are usually returned by qz.dgees.

There is also no checking for select which is usually according to the returns of qz.dgeev.

### Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

### References

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/double/dtrsens.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

### See Also

[qz.zgees](#), [qz.dgees](#), [qz.ztrsens](#).

### Examples

```
library(QZ, quiet = TRUE)

### https://www.nag.com/numeric/fl/nagdoc_f122/xhtmll/f08/f08qgf.xml
T <- exA4$T
Q <- exA4$Q
select <- c(TRUE, FALSE, FALSE, TRUE)
ret <- qz.dtrsens(T, Q, select)

# Verify 1
A <- Q %*% T %*% solve(Q)
A.new <- ret$Q %*% ret$T %*% solve(ret$Q)
round(A - A.new)

# verify 2
round(ret$Q %*% t(ret$Q))
```



---

qz.zgees

*QZ Decomposition for a Complex Matrix*


---

**Description**

This function call 'zgees' in Fortran to decompose a 'complex' matrix A.

**Usage**

```
qz.zgees(A, vs = TRUE, LWORK = NULL)
```

**Arguments**

A	a 'complex' matrix, dim = c(N, N).
vs	if compute 'complex' Schur vectors. (Q)
LWORK	optional, dimension of array WORK for workspace. (>= 2N)

**Details**

See 'zgees.f' for all details.

ZGEEES computes for an N-by-N complex non-symmetric matrix A, the eigenvalues, the Schur form T, and, optionally, the matrix of Schur vectors Q. This gives the Schur factorization  $A = Q * T * (Q^{*} * H)$ .

Optionally, it also orders the eigenvalues on the diagonal of the Schur form so that selected eigenvalues are at the top left. The leading columns of Q then form an orthonormal basis for the invariant subspace corresponding to the selected eigenvalues.

A complex matrix is in Schur form if it is upper triangular.

**Value**

Return a list contains next:

'T'	A's generalized Schur form.
'W'	generalized eigenvalues.
'VS'	original returns from 'zgees.f'.
'WORK'	optimal LWORK (for zgees.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1,...,N: QZ iteration failed. =N+1: reordering problem. =N+2: reordering failed.

Extra returns in the list:

'Q'	the Schur vectors.
-----	--------------------

**Warning(s)**

The results may not be consistent on 32 bits and 64 bits Windows systems, but may be valid on both systems.

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.  
<https://www.netlib.org/lapack/complex16/zgees.f>  
[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

[qz.zgeev](#)

**Examples**

```
library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node94.html
A <- exA1$A
ret <- qz.zgees(A)

# Verify 1
A.new <- ret$Q %*% ret$T %*% H(ret$Q)
round(A - A.new)

# verify 2
round(ret$Q %*% H(ret$Q))
```

---

qz.zgeev

*Generalized Eigenvalues Decomposition for a Complex Matrix*

---

**Description**

This function call 'zgeev' in Fortran to decompose a 'complex' matrix A.

**Usage**

```
qz.zgeev(A, v1 = TRUE, vr = TRUE, LWORK = NULL)
```

**Arguments**

A	a 'complex' matrix, dim = c(N, N).
v1	if compute left 'complex' eigen vectors. (U)
vr	if compute right 'complex' eigen vectors. (V)
LWORK	optional, dimension of array WORK for workspace. (>= 2N)

**Details**

See 'zgeev.f' for all details.

ZGEEV computes for an N-by-N complex non-symmetric matrix A, the eigenvalues and, optionally, the left and/or right eigenvectors.

The right eigenvector v(j) of A satisfies

$$A * v(j) = \text{lambda}(j) * v(j)$$

where lambda(j) is its eigenvalue. The left eigenvector u(j) of A satisfies

$$u(j)**H * A = \text{lambda}(j) * u(j)**H$$

where u(j)\*\*H denotes the conjugate transpose of u(j).

The computed eigenvectors are normalized to have Euclidean norm equal to 1 and largest component real.

**Value**

Return a list contains next:

'W'	original returns from 'zgeev.f'.
'VL'	original returns from 'zgeev.f'.
'VR'	original returns from 'zgeev.f'.
'WORK'	optimal LWORK (for zgeev.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. > 0: QZ iteration failed.

Extra returns in the list:

'U'	the left eigen vectors.
'V'	the right eigen vectors.

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/complex16/zgeev.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

[qz.zgees](#)

**Examples**

```

library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node92.html
A <- exA1$A
ret <- qz.zggev(A)

# Verify 1
diff.R <- A %% ret$V - matrix(ret$W, 4, 4, byrow = TRUE) * ret$V
diff.L <- H(ret$U) %% A - matrix(ret$W, 4, 4) * H(ret$U)
round(diff.R)
round(diff.L)

# Verify 2
round(ret$U %% H(ret$U))
round(ret$V %% H(ret$V))

```

qz.zgges

*QZ Decomposition for Complex Paired Matrices***Description**

This function call 'zgges' in Fortran to decompose 'complex' matrices (A,B).

**Usage**

```
qz.zgges(A, B, vs1 = TRUE, vsr = TRUE, LWORK = NULL)
```

**Arguments**

A	a 'complex' matrix, dim = c(N, N).
B	a 'complex' matrix, dim = c(N, N).
vs1	if compute left 'complex' Schur vectors. (Q)
vsr	if compute right 'complex' Schur vectors. (Z)
LWORK	optional, dimension of array WORK for workspace. (>= 2N)

**Details**

See 'zgges.f' for all details.

ZGGES computes for a pair of N-by-N complex non-symmetric matrices (A,B), the generalized eigenvalues, the generalized complex Schur form (S, T), and optionally left and/or right Schur vectors (VSL and VSR). This gives the generalized Schur factorization

$$(A,B) = ( (VSL)*S*(VSR)**H, (VSL)*T*(VSR)**H )$$

where (VSR)\*\*H is the conjugate-transpose of VSR.

Optionally, it also orders the eigenvalues so that a selected cluster of eigenvalues appears in the leading diagonal blocks of the upper triangular matrix S and the upper triangular matrix T. The leading columns of VSL and VSR then form an unitary basis for the corresponding left and right eigenspaces (deflating subspaces).

(If only the generalized eigenvalues are needed, use the driver ZGGEV instead, which is faster.)

A generalized eigenvalue for a pair of matrices (A,B) is a scalar  $w$  or a ratio  $\alpha/\beta = w$ , such that  $A - w*B$  is singular. It is usually represented as the pair (alpha,beta), as there is a reasonable interpretation for  $\beta=0$ , and even for both being zero.

A pair of matrices (S,T) is in generalized complex Schur form if S and T are upper triangular and, in addition, the diagonal elements of T are non-negative real numbers.

### Value

Return a list contains next:

'S'	A's generalized Schur form.
'T'	B's generalized Schur form.
'ALPHA'	ALPHA[j]/BETA[j] are generalized eigenvalues.
'BETA'	ALPHA[j]/BETA[j] are generalized eigenvalues.
'VSL'	original returns from 'zgges.f'.
'VSR'	original returns from 'zgges.f'.
'WORK'	optimal LWORK (for zgges.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1,...,N: QZ iteration failed. =N+1: other than QZ iteration failed in ZHGEQZ. =N+2: reordering problem. =N+3: reordering failed.

Extra returns in the list:

'Q'	the left Schur vectors.
'Z'	the right Schur vectors.

### Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

### References

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/complex16/zgges.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

### See Also

[qz.zggeev](#)

**Examples**

```

library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node124.html
A <- exAB1$A
B <- exAB1$B
ret <- qz.zgges(A, B)

# Verify 1
A.new <- ret$Q %*% ret$S %*% H(ret$Z)
B.new <- ret$Q %*% ret$T %*% H(ret$Z)
round(A - A.new)
round(B - B.new)

# verify 2
round(ret$Q %*% H(ret$Q))
round(ret$Z %*% H(ret$Z))

```

---

qz.zggeev

*Generalized Eigenvalues Decomposition for Complex Paired Matrices*


---

**Description**

This function call 'zggev' in Fortran to decompose 'complex' matrices (A,B).

**Usage**

```
qz.zggeev(A, B, vl = TRUE, vr = TRUE, LWORK = NULL)
```

**Arguments**

A	a 'complex' matrix, dim = c(N, N).
B	a 'complex' matrix, dim = c(N, N).
vl	if compute left 'complex' eigen vectors. (U)
vr	if compute right 'complex' eigen vectors. (V)
LWORK	optional, dimension of array WORK for workspace. (>= 2N)

**Details**

See 'zggev.f' for all details.

ZGGEV computes for a pair of N-by-N complex non-symmetric matrices (A,B), the generalized eigenvalues, and optionally, the left and/or right generalized eigenvectors.

A generalized eigenvalue for a pair of matrices (A,B) is a scalar lambda or a ratio alpha/beta = lambda, such that A - lambda\*B is singular. It is usually represented as the pair (alpha,beta), as there is a reasonable interpretation for beta=0, and even for both being zero.

The right generalized eigenvector  $v(j)$  corresponding to the generalized eigenvalue  $\lambda(j)$  of  $(A,B)$  satisfies

$$A * v(j) = \lambda(j) * B * v(j).$$

The left generalized eigenvector  $u(j)$  corresponding to the generalized eigenvalues  $\lambda(j)$  of  $(A,B)$  satisfies

$$u(j)**H * A = \lambda(j) * u(j)**H * B$$

where  $u(j)**H$  is the conjugate-transpose of  $u(j)$ .

## Value

Return a list contains next:

'ALPHA'	original returns from 'zggev.f'.
'BETA'	original returns from 'zggev.f'.
'VL'	original returns from 'zggev.f'.
'VR'	original returns from 'zggev.f'.
'WORK'	optimal LWORK (for zggev.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1,...,N: QZ iteration failed. =N+1: other than QZ iteration failed in ZHGEQZ. =N+2: reordering problem. =N+3: reordering failed.

Extra returns in the list:

'U'	the left eigen vectors.
'V'	the right eigen vectors.

Note that 'VL' and 'VR' are scaled so the largest component has  $\text{abs}(\text{real part}) + \text{abs}(\text{imag. part}) = 1$ .

## Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

## References

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/complex16/zggeev.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

## See Also

[qz.zgges](#)

## Examples

```

library(QZ, quiet = TRUE)

### https://www.nag.com/lapack-ex/node122.html
A <- exAB1$A
B <- exAB1$B
ret <- qz.zggeev(A, B)

# Verify 1
(lambda <- ret$ALPHA / ret$BETA) # Unstable
diff.R <- matrix(ret$BETA, 4, 4, byrow = TRUE) * A %% ret$V -
          matrix(ret$ALPHA, 4, 4, byrow = TRUE) * B %% ret$V
diff.L <- matrix(ret$BETA, 4, 4) * H(ret$U) %% A -
          matrix(ret$ALPHA, 4, 4) * H(ret$U) %% B
round(diff.R)
round(diff.L)

# Verify 2
round(ret$U %% solve(ret$U))
round(ret$V %% solve(ret$V))

```

---

qz.ztgsen

*Reordered QZ Decomposition for Complex Paired Matrices*


---

## Description

This function call 'ztgsend' in Fortran to reorder 'complex' matrices (S,T,Q,Z).

## Usage

```

qz.ztgsen(S, T, Q, Z, select, ijob = 4L,
          want.Q = TRUE, want.Z = TRUE, LWORK = NULL, LIWORK = NULL)

```

## Arguments

S	a 'complex' generalized Schur form, dim = c(N, N).
T	a 'complex' generalized Schur form, dim = c(N, N).
Q	a 'complex' left Schur vectors, dim = c(N, N).
Z	a 'complex' right Schur vectors, dim = c(N, N).
select	specifies the eigenvalues in the selected cluster.
ijob	specifies whether condition numbers are required for the cluster of eigenvalues (PL and PR) or the deflating subspaces (Difu and Difl).
want.Q	if update Q.
want.Z	if update Z.
LWORK	optional, dimension of array WORK for workspace. ( $\geq N(N+1)$ )
LIWORK	optional, dimension of array IWORK for workspace. ( $\geq \max(N+2, N(N+1)/2)$ )



## Details

See 'ztgsen.f' for all details.

ZTGSEN reorders the generalized Schur decomposition of a complex matrix pair (S,T) (in terms of an unitary equivalence transformation  $Q^{**H} * (S,T) * Z$ ), so that a selected cluster of eigenvalues appears in the leading diagonal blocks of the pair (S,T). The leading columns of Q and Z form unitary bases of the corresponding left and right eigenspaces (deflating subspaces). (S,T) must be in generalized Schur canonical form, that is, S and T are both upper triangular.

ZTGSEN also computes the generalized eigenvalues

$$w(j) = \text{ALPHA}(j) / \text{BETA}(j)$$

of the reordered matrix pair (S,T).

Note for 'ijob':

=0: Only reorder w.r.t. SELECT. No extras.

=1: Reciprocal of norms of "projections" onto left and right eigenspaces w.r.t. the selected cluster (PL and PR).

=2: Upper bounds on DifU and DifL. F-norm-based estimate (DIF(1:2)).

=3: Estimate of DifU and DifL. 1-norm-based estimate (DIF(1:2)). About 5 times as expensive as ijob = 2.

=4: Compute PL, PR and DIF (i.e. 0, 1 and 2 above): Economic version to get it all.

=5: Compute PL, PR and DIF (i.e. 0, 1 and 3 above).

In short, if  $(A,B) = Q * (S,T) * Z^{**H}$  from qz.zgges and input (S,T,Q,Z) to qz.ztgsen with appropriate select option, then it yields

$$(A,B) = Q\_n * (S\_n, T\_n) * Z\_n^{**H}$$

where  $(S\_n, T\_n, Q\_n, Z\_n)$  is a new set of generalized Schur decomposition of (A,B) according to the select.

## Value

Return a list contains next:

'S'	S's reorded generalized Schur form.
'T'	T's reorded generalized Schur form.
'ALPHA'	ALPHA[j]/BETA[j] are generalized eigenvalues.
'BETA'	ALPHA[j]/BETA[j] are generalized eigenvalues.
'M'	original returns from 'ztgsen.f'.
'PL'	original returns from 'ztgsen.f'.
'PR'	original returns from 'ztgsen.f'.
'DIF'	original returns from 'ztgsen.f'.
'WORK'	optimal LWORK (for ztgsen.f only)
'IWORK'	optimal LIWORK (for ztgsen.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value. =1: reordering of (S,T) failed.

Extra returns in the list:

'Q'                    the reorded left Schur vectors.  
'Z'                    the reorded right Schur vectors.

### Warning(s)

There is no format checking for S, T, Q, and Z which are usually returned by qz.zgges.

There is also no checking for select which is usually according to the returns of qz.zggev.

### Author(s)

Wei-Chen Chen <wccsnow@gmail.com>

### References

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/complex16/ztgsen.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

### See Also

[qz.zgges](#), [qz.dgges](#), [qz.dtgsen](#).

### Examples

```
library(QZ, quiet = TRUE)

### https://www.nag.com/numeric/fl/nagdoc_fl123/xhtml/f08/f08yuf.xml
S <- exAB3$S
T <- exAB3$T
Q <- exAB3$Q
Z <- exAB3$Z
select <- c(FALSE, TRUE, TRUE, FALSE)
ret <- qz.ztgsen(S, T, Q, Z, select)

# Verify 1
S.new <- ret$Q %*% ret$S %*% H(ret$Z)
T.new <- ret$Q %*% ret$T %*% H(ret$Z)
round(S - S.new)
round(T - T.new)

# verify 2
round(ret$Q %*% H(ret$Q))
round(ret$Z %*% H(ret$Z))
```

qz.ztrsen

*Reordered QZ Decomposition for a Complex Matrix***Description**

This function call 'ztrsend' in Fortran to reorder 'complex' matrix (T,Q).

**Usage**

```
qz.ztrsen(T, Q, select, job = c("B", "V", "E", "N"),
          want.Q = TRUE, LWORK = NULL)
```

**Arguments**

T	a 'complex' generalized Schur form, dim = c(N, N).
Q	a 'complex' Schur vectors, dim = c(N, N).
select	specifies the eigenvalues in the selected cluster.
job	Specifies whether condition numbers are required for the cluster of eigenvalues (S) or the invariant subspace (SEP).
want.Q	if update Q.
LWORK	optional, dimension of array WORK for workspace. ( $\geq N(N+1)/2$ )

**Details**

See 'ztrsen.f' for all details.

ZTRSEN reorders the Schur factorization of a complex matrix  $A = Q^*T^*Q^{**}H$ , so that a selected cluster of eigenvalues appears in the leading positions on the diagonal of the upper triangular matrix T, and the leading columns of Q form an orthonormal basis of the corresponding right invariant subspace.

Optionally the routine computes the reciprocal condition numbers of the cluster of eigenvalues and/or the invariant subspace.

**Value**

Return a list contains next:

'T'	T's reorded generalized Schur form.
'W'	generalized eigenvalues.
'M'	original returns from 'ztrsen.f'.
'S'	original returns from 'ztrsen.f'.
'SEP'	original returns from 'ztrsen.f'.
'WORK'	optimal LWORK (for ztrsen.f only)
'INFO'	= 0: successful. < 0: if INFO = -i, the i-th argument had an illegal value.

Extra returns in the list:

'Q'	the reorded Schur vectors.
-----	----------------------------

**Warning(s)**

There is no format checking for T and Q which are usually returned by qz.zgees.

There is also no checking for select which is usually according to the returns of qz.zgeev.

**Author(s)**

Wei-Chen Chen <wccsnow@gmail.com>

**References**

Anderson, E., et al. (1999) *LAPACK User's Guide*, 3rd edition, SIAM, Philadelphia.

<https://www.netlib.org/lapack/complex16/ztrsen.f>

[https://en.wikipedia.org/wiki/Schur\\_decomposition](https://en.wikipedia.org/wiki/Schur_decomposition)

**See Also**

[qz.zgees](#), [qz.dgees](#), [qz.dtrsen](#).

**Examples**

```
library(QZ, quiet = TRUE)

### https://www.nag.com/numeric/f1/nagdoc_f123/xhtml/f08/f08quf.xml
T <- exA3$T
Q <- exA3$Q
select <- c(TRUE, FALSE, FALSE, TRUE)
ret <- qz.ztrsen(T, Q, select)

# Verify 1
A <- Q %*% T %*% solve(Q)
A.new <- ret$Q %*% ret$T %*% solve(ret$Q)
round(A - A.new)

# verify 2
round(ret$Q %*% solve(ret$Q))
```

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